

Corrections to “Optimal Energy Shaping and Force Amplification Framework for Task-Agnostic, Biomimetic Ankle Exoskeletons”

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This note addresses unstated assumptions and errors in the proof of Theorem 1 in [1]. The correctness of Theorem 1 relies on two additional assumptions that were not stated in the original manuscript: that the human gravity compensation in the impedance model (10) compensates the shaped potential rather than the gravitational potential alone, and that the controller’s velocity-dependent shaping vanishes at the equilibrium configuration, which holds for the configurations used in the experimental stability evaluation. Independent of these assumptions, the proof in the Appendix requires correction in three places: a scalar sufficient condition, a derivation step that relies on the inverse of a rank-deficient matrix, and the forward-invariance argument. Below we state the assumptions enabling the original Theorem 1, give a generalized statement of Theorem 1 that removes the second assumption, and provide a corrected proof for the general case. No experimental results, figures, or conclusions of [1] are affected.

REFERENCES

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I. UNSTATED ASSUMPTIONS AND GENERALIZED THEOREM 1

The correctness of Theorem 1 in [1] relies on two unstated assumptions. Both concern the local linearization of the closed-loop dynamics (2) under the impedance-controlled human input (10).

Assumption 1 (shaped gravity compensation). The gravity compensation term $G(\bar{q})$ in (10) compensates the shaped potential $\tilde{V} = V + \hat{V}$ rather than the gravitational potential V alone:

$$G(\bar{q}) = (\partial_q \tilde{V})^T|_{q=\bar{q}}, \quad (C1)$$

i.e., the human provides the constant effort required to hold the setpoint $\bar{q}$ against gravity and the constant shaping torque at $\bar{q}$. Under this definition, $\bar{q}$ is an equilibrium of (2) under (10), and $\nabla G(q)|_{q=\bar{q}} = \nabla^2 \tilde{V}(\bar{q})$ in (11) contains the stiffness of the shaped potential, so the total effective spring rate K_p defined in (11) is unchanged. As in [1], we require $K_p > 0$; because the Hessian of $\tilde{V}$ is linear in the optimized coefficients, this requirement combines a verifiable design property of the controller with an assumption on the human impedance.

Assumption 2 (velocity-dependent shaping at the equilibrium). The interconnection term $J_2(\partial_p \tilde{H})^T$ in (2) contributes to the linearized damping of the closed-loop system; this contribution is not represented in the state-space model (21) of [1], and vanishes under the following assumption. Let

$$\bar{J} = J_2(\bar{q}), \quad \bar{\kappa} = \kappa(\bar{q}), \quad (C2)$$

denote the velocity-dependent shaping matrix and the scalar parameterizing it at the human setpoint, per Sec. III of [1]. The model (21) and condition (13) hold as published when $\bar{J} = 0$, i.e., when $\bar{\kappa} = 0$. For the basis functions $\mathcal{B}_\psi = [\psi, \sin\psi, \sin 2\psi]$ used in [1], $\bar{\kappa} = 0$ at any equilibrium with a vertical shank ($\bar{\psi} = 0$), which includes the configurations used in the experimental stability evaluation.

The following generalization of Theorem 1 removes Assumption 2. It holds locally around arbitrary $\bar{q}$, where the mass/inertia matrix M is approximately constant.

Theorem 1 (generalized). Assume the mass/inertia matrix M is constant and that $G(\bar{q})$ satisfies (C1). The system (1) under exoskeleton controller (8) and human spring-damper behavior (10) satisfying (12) and

$$\alpha \left(4M + (K_d - \bar{J})^T K_p^{-1} (K_d - \bar{J}) \right) < 4K_d, \quad (C3)$$

with scalar $\alpha > 0$, is stable to an invariant set—a locally attractive set of states within which the system will remain—bounded by a function of the magnitude of the control inputs λ_F and u_x , acting on the system through external ports (9).

The corrected proof in Sec. III addresses this generalized form of Theorem 1. Note that the right side of (C3) remains $4K_d$ because the skew-symmetric $\bar{J}$ contributes no damping; it enters only through the coupling term on the left side. Condition (C3) reduces to (13) of [1] when $\bar{J} = 0$.

II. CORRECTIONS TO THE PUBLISHED PROOF

Independently of the assumptions above, the proof in the Appendix of [1] requires the following corrections, which are incorporated in the generalized proof of Sec. III.

(a) In the scalar sufficient condition (20), $\min_\lambda(K_p)$ should divide rather than multiply; the Schur complement step yields $(\max_\lambda(K_d))^2 / \min_\lambda(K_p)$ in the denominator. The corrected condition, which also accounts for $\bar{J}$, is (C5) below.

(b) Equations (31)–(35) substitute $y = Qx$ with $QQ = \bar{Q}$ and use Q^{-1} . Because $\bar{Q}$ has rank n in $2n$ dimensions, Q is singular and its inverse does not exist. The corrected derivation, (C15)–(C19), bounds both sides of (C14) by extreme eigenvalues in the original coordinates; Lemma 1 of [1] is no longer needed.

(c) The forward invariance of (35) was inferred from $\dot{\mathcal{W}} \leq 0$ on its boundary. That inference requires the boundary to be a level set of $\mathcal{W}$, which the boundary of the quadric (35) is not. The corrected invariant set (C19) is a sublevel set of $\mathcal{W}$, for which this inference holds, admitting an explicit ultimate bound on $\|x\|$.

The corrected proof also states the disturbance bound (29) as an equality, since the Cauchy–Schwarz bound is attained, and addresses the contact constraint forces following (C6).

III. CORRECTED PROOF OF THEOREM 1 (GENERALIZED)

We define functions $\max_\lambda(A)$ and $\min_\lambda(A)$ to return the maximum or minimum eigenvalue, respectively, of a matrix $A \in \mathbb{R}^{n \times n}$. Equation numbers without the prefix C refer to [1].

Proof (generalized). The assumptions (12) and (C3) are matrix conditions; we first verify their feasibility by exhibiting scalar sufficient conditions. Any α satisfying

$$\alpha < \sqrt{\frac{\min_\lambda(K_p)}{\max_\lambda(M)}} \quad (C4)$$

and

$$\alpha < \frac{4\min_\lambda(K_d)}{4\max_\lambda(M) + (\max_\lambda(K_d) + |\bar{\kappa}|)^2 / \min_\lambda(K_p)} \quad (C5)$$

also satisfies (12) and (C3). The first implication holds because $\alpha^2 \max_\lambda(M) < \min_\lambda(K_p)$ implies $\alpha^2 M < K_p$. The second holds because $(K_d - \bar{J})^T K_p^{-1} (K_d - \bar{J}) \leq \max_\lambda(K_p^{-1})(\max_\lambda(K_d) + |\bar{\kappa}|)^2 I$, as the nonzero singular values of $\bar{J}$ equal $|\bar{\kappa}|$. Both bounds admit a small positive α .

To analyze system stability, we define the state-space model with error state $x = [e^T, \dot{e}^T]^T$, where $e = q - \bar{q}$ as in (10), and system dynamics

$$\dot{x} = Ax + \Lambda, \quad A = \begin{bmatrix} 0 & I \\ -M^{-1}K_p & -M^{-1}(K_d - \bar{J}) \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 0 \\ \delta \end{bmatrix}, \quad (C6)$$

and $\delta = M^{-1}(\hat{A}^T \lambda_F + B_x u_x)$ expresses the non-passive port inputs (GRF shaping and power leak) as an external disturbance. Here, (C6) linearizes the target system (2) under the human input (10) about $\bar{q}$, where the interconnection term contributes the skew-symmetric $\bar{J}$ to the damping block, and higher-order terms are neglected. The workless constraint forces $A^T \lambda_F$ are omitted. During stance, both the trajectory and the setpoint satisfy the contact constraint, $a(q) = 0$ and $a(\bar{q}) = 0$, so expanding about $\bar{q}$ gives $A(\bar{q})e = 0$ and $A(\bar{q})\dot{e} = 0$ to first order. The error thus evolves in the null space of $A(\bar{q})$, orthogonal to the constraint forces; the swing phase has no contact constraints.

Let $\beta \geq \delta^T \delta$, where $\beta \in \mathbb{R}^+$ is a bound on the quadratic disturbance. We assume that software limits saturate the GRFs such that the shaping term $\hat{A}^T \lambda_F$ is bounded. The power leak term $B_x u_x$ is bounded by the bounded basis functions (Sec. III of [1]) under biomechanically bounded velocity magnitudes.

Consider the following candidate Lyapunov function,

$$\mathcal{W}(x) = \frac{1}{2} x^T W x, \quad W = \begin{bmatrix} K_p & \alpha M \\ \alpha M & M \end{bmatrix}. \quad (C7)$$

We establish positive-definiteness from the Schur complement:

$$W > 0 \Leftrightarrow \begin{cases} M > 0, \\ K_p - \alpha M M^{-1} \alpha M > 0, \end{cases} \quad (C8)$$

with the first case true by positive-definiteness of the mass/inertia matrix and the second equivalent to the assumption (12). The function $\mathcal{W}$ has time-derivative

$$\dot{\mathcal{W}} = \frac{1}{2} x^T (WA + A^T W)x + x^T W \Lambda = \frac{1}{2} x^T \bar{W} x + x^T W \Lambda, \quad (C9)$$

where

$$\bar{W} = \begin{bmatrix} -2\alpha K_p & -\alpha(K_d - \bar{J}) \\ -\alpha(K_d + \bar{J}) & 2\alpha M - 2K_d \end{bmatrix}. \quad (C10)$$

Here, $\bar{J}$ canceled in the lower-right block, as $\bar{J} + \bar{J}^T = 0$.

Given the linearizing assumptions with constant matrices, we will show $\bar{W} < 0$, which will confine non-negative $\dot{\mathcal{W}}$ to a bounded region of the state space. By the Schur complement, the negative definiteness of $\bar{W}$ is equivalent to

$$\begin{cases} -2\alpha K_p < 0, \\ 2\alpha M - 2K_d + \alpha(K_d + \bar{J})(2\alpha K_p)^{-1} \alpha(K_d - \bar{J}) < 0, \end{cases} \quad (C11)$$

with the first true by the assumption that $K_p > 0$ and $\alpha > 0$, and the second equivalent to assumption (C3), using the identity $(K_d + \bar{J})K_p^{-1}(K_d - \bar{J}) = (K_d - \bar{J})^T K_p^{-1}(K_d - \bar{J}) \geq 0$, which is symmetric. With $\bar{W} < 0$, we now characterize the region of the state space in which $\dot{\mathcal{W}} \leq 0$ is guaranteed.

A sufficient condition for $\dot{\mathcal{W}} \leq 0$ is

$$\frac{1}{2} x^T \bar{W} x \leq -\max_{\delta} x^T W \Lambda, \quad (C12)$$

where the right side represents the worst-case increase in $\dot{\mathcal{W}}$ resulting from a maximum value of the disturbance, δ . Because the scalar $x^T W \Lambda = \delta^T [\alpha M; M]^T x$ is linear in δ , its maximum over $\delta^T \delta \leq \beta$ is attained with equality by Cauchy–Schwarz,

$$\max_{\delta} x^T W \Lambda = \sqrt{x^T \bar{Q} x}, \quad \bar{Q} = \begin{bmatrix} \alpha M \\ M \end{bmatrix} \beta \begin{bmatrix} \alpha M \\ M \end{bmatrix}^T \geq 0, \quad (C13)$$

so (C12) is equivalent to

$$\frac{1}{2}x^T\overline{W}x \leq -\sqrt{x^T\overline{Q}x}. \quad (\text{C14})$$

Note that $x^T\overline{Q}x = \beta \| \alpha M e + M \dot{e} \|^2$ vanishes on the subspace $\dot{e} = -\alpha e$, where the disturbance $x^T W \Lambda = \delta^T [\alpha M ; M]^T x$ also vanishes, so (C14) holds there by $\overline{W} < 0$ alone.

Away from this subspace we bound each side by its extreme eigenvalue. The negative-definite matrix $\overline{W}$ gives

$$\frac{1}{2}x^T\overline{W}x \leq \frac{1}{2}\max_{\lambda}(\overline{W}) \| x \|^2, \quad (\text{C15})$$

and the disturbance magnitude is bounded by

$$\sqrt{x^T\overline{Q}x} \leq \sqrt{\max_{\lambda}(\overline{Q})} \| x \|. \quad (\text{C16})$$

Substituting (C15) and (C16) into (C14), we obtain

$$\frac{1}{2}\max_{\lambda}(\overline{W}) \| x \|^2 \leq -\sqrt{\max_{\lambda}(\overline{Q})} \| x \|. \quad (\text{C17})$$

The nonzero eigenvalues of $\overline{Q}$ equal those of $\beta(1 + \alpha^2)M^2$, so $\max_{\lambda}(\overline{Q}) = \beta(1 + \alpha^2)(\max_{\lambda}(M))^2$. Simplifying in terms of $\| x \|$ then yields $\| x \| \geq r$ with

$$r := \frac{2\sqrt{\beta(1 + \alpha^2)}\max_{\lambda}(M)}{\min_{\lambda}(-\overline{W})}, \quad (\text{C18})$$

determined by the maximum bound on the disturbance (β) and the minimum human impedance (captured in $\overline{W}$).

Thus, for any disturbance satisfying $\delta^T \delta \leq \beta$, $\dot{\mathcal{W}} < 0$ wherever $\| x \| > r$, with $\dot{\mathcal{W}} \leq 0$ at $\| x \| = r$. As the level sets of $\mathcal{W}$ are ellipsoids, the ball $\{\| x \| \leq r\}$ itself need not be invariant. We therefore take the invariant set to be the smallest sublevel set of $\mathcal{W}$ containing this ball,

$$\mathcal{R} = \left\{ x : \mathcal{W}(x) \leq \frac{1}{2}\max_{\lambda}(W) r^2 \right\}. \quad (\text{C19})$$

Since $\mathcal{W}(x) \leq \frac{1}{2}\max_{\lambda}(W) \| x \|^2$, any x outside $\mathcal{R}$ satisfies $\| x \| > r$, with $\| x \| \geq r$ on its boundary, so by the sign of $\dot{\mathcal{W}}$ no trajectory exits $\mathcal{R}$: it is forward invariant [2]. By (C15) and (C16), $\dot{\mathcal{W}}$ is bounded above outside $\mathcal{R}$, uniformly in the disturbance, by a continuous function of $\| x \|$ that is strictly negative for $\| x \| > r$, so trajectories converge to $\mathcal{R}$ by a standard ultimate boundedness argument [3, Theorem 4.18]. Finally, $\frac{1}{2}\min_{\lambda}(W) \| x \|^2 \leq \mathcal{W}(x)$ on $\mathcal{R}$ yields the ultimate bound $\| x \| \leq r\sqrt{\max_{\lambda}(W)/\min_{\lambda}(W)}$, determined by β and the minimum human impedance. ▀